Scalar Fields Nonminimally Coupled to *pp* Waves

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Here, we report *pp* waves configurations of three-dimensional gravity for which a scalar field nonminimally coupled to them acts as a source. In absence of self-interaction the solutions are gravitational plane waves with a profile fixed in terms of the scalar wave. In the self-interacting case, only power-law potentials parameterized by the nonminimal coupling constant are allowed by the field equations. In contrast with the free case the self-interacting scalar field does not behave like a wave since it depends only on the wave-front coordinate. We address the same problem when gravitation is governed by topologically massive gravity and the source is a free scalar field. From the *pp* waves derived in this case, we obtain at the zero topological mass limit, new *pp* wave solutions of conformal gravity for any arbitrary value of the nonminimal coupling parameter. Finally, we extend these solutions to the self-interacting case of conformal gravity.

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I. INTRODUCTION

Recently, it has been shown that a version of three-dimensional gravity governed just by the Cotton tensor with a conformally invariant free source given by a scalar field admits *pp* wave solutions [1]. The same model with a conformal self-interacting potential also supports such gravitational fields [2]. In view of these two works, a natural question is to ask whether *pp* wave configurations are also solutions to standard three-dimensional Einstein gravity with scalar source. This problem is of interest since it is well known that pure gravitational waves are forbidden in three dimensions. In contrast with these previous works, since the Einstein tensor is not conformally invariant, this allows to consider a scalar field nonminimally coupled to gravity for which the coupling parameter is not necessarily the conformal one. The three-dimensional action we consider here is given by

$$S(g_{\alpha\beta}, \Phi) = \int d^3x \sqrt{-g} \left(\frac{1}{2\kappa} R - \frac{1}{2} \nabla_\alpha \Phi \nabla^\alpha \Phi - \frac{1}{2} \xi R \, \Phi^2 - U(\Phi) \right), \qquad (1)$$

where ξ is the nonminimal coupling parameter and $U(\Phi)$ is the self-interaction potential. The field equations obtained by varying the metric (resp. the scalar field) read

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}, \qquad (2)$$

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and

$$\Box\Phi = \xi R\,\Phi + \frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}, \tag{3}$$

where the energy-momentum tensor is given by

$$T_{\alpha\beta} = \nabla_\alpha\Phi\nabla_\beta\Phi - g_{\alpha\beta}\left(\frac{1}{2}\nabla_\sigma\Phi\nabla^\sigma\Phi + U(\Phi)\right) + \xi\left(g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta + G_{\alpha\beta}\right)\Phi^2. \tag{4}$$

In what follows we shall explore the existence of pp wave configurations for the previous system. In the next section, we present the independent field equations for a pp wave background in three dimensions. Sec. III is devoted to the free case (i.e., $U(\Phi) = 0$) for which it is shown that the solutions are gravitational plane waves whose profiles are determined by the wave profile of the free scalar field. The analysis of the self-interacting case is done in Sec. IV. In this set-up, only power-law potentials are allowed by the field equations where the power is given in terms of the nonminimal coupling parameter. In contrast with the free situation, the scalar field does not behave like a wave in accordance with its self-interacting nature. Moreover, for a nonminimal coupling parameter $\xi = 1/2$, the potential reduces to a positive constant which allows to interpret the solutions as pp waves for a free scalar field when the Einstein equations are supplemented by a positive cosmological constant. In Sec. V, we present the pp wave solutions of the topologically massive gravity when the scalar field is free. The related gravitational waves exhibit an effective mass given in terms of the nonminimal coupling parameter. In Sec. VI, we show that at the zero topological mass limit, the above solutions turn out to be pp wave solutions of conformal gravity for any arbitrary value of the nonminimal coupling parameter. This result is a generalization of the conformal pp waves of Ref. [1] since the energy-momentum tensor of these solutions evaluated on-shell is traceless independently of the value of the nonminimal coupling parameter. Taking into consideration this last remark, we have also derived in the same section, the pp wave configurations of conformal gravity with a self-interacting nonminimally scalar field. Finally, the appendices are devoted to the analysis of some special values of the nonminimal coupling parameter as for example the case $\xi = 1/4$ which requires a separate derivation and for which the allowed configuration corresponds to a free massive scalar field.

II.   *PP* WAVE FIELD EQUATIONS

The term pp wave is an abbreviation for plane-fronted gravitational waves with parallel rays, which are the gravitational configurations possessing a covariantly constant null vector field [3]. The corresponding geometry is written in three dimensions as

$$ds^2 = -F(u,y)du^2 - 2dudv + dy^2, \tag{5}$$

where the covariantly constant null vector field is ∂_v. In three dimensions, the front of the wave (the surfaces at constant u, v) is just a line and not a plane as it occurs in higher dimensions, and hence it would be more appropriate to call the geometries (5) line fronted gravitational waves. However, in order to avoid any confusions, we shall use the term pp wave as usual.

The null field ∂_v is a Killing field and so we impose the same symmetry on the source, that means $\Phi = \Phi(u,y)$. For the geometry (5), the only nonvanishing component of the Einstein tensor is G_{uu}. Consequently, all the components of the energy-momentum tensor

except T_{uu} must vanish by virtue of the Einstein equations. For convenience, we choose the following combinations of this tensor

$$T_{uv} + T_{yy} = (1 - 2\xi)(\partial_y\Phi)^2 - 2\xi\Phi\partial_{yy}^2\Phi = 0, \tag{6}$$

$$T_{uy} = (1 - 2\xi)\partial_u\Phi\partial_y\Phi - 2\xi\Phi\partial_{uy}^2\Phi = 0, \tag{7}$$

$$T_{yy} = \frac{1}{2}(\partial_y\Phi)^2 - U(\Phi) = 0, \tag{8}$$

while the remaining independent Einstein equation can be taken as

$$G_{uu} - \kappa(T_{uu} - FT_{uv}) = \frac{1}{2}(1 - \kappa\xi\Phi^2)\partial_{yy}^2F - \kappa\xi\Phi(\partial_y\Phi\partial_yF - 2\partial_{uu}^2\Phi) - \kappa(1 - 2\xi)(\partial_u\Phi)^2 = 0. \tag{9}$$

The system of Eqs. (6)-(9) are all the independent Einstein equations and in what follows we shall always refer to these equations. We do not take into consideration the scalar equation (3) which takes the form

$$\partial_{yy}^2\Phi = \frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}, \tag{10}$$

since the conservation of the energy-momentum tensor (4) together with the existence of a nontrivial scalar field solution of the Einstein equations (2) guarantee that this equation is satisfied.

In the next section, we analyze the free case, $U(\Phi) = 0$, for which the solutions turn out to be plane waves, i.e., where the metric dependence on the front-wave coordinate y is just quadratic.

III. FREE SCALAR FIELDS: PLANE WAVES

In this section, we consider a scalar field nonminimally coupled to gravity without self-interaction (i.e., $U(\Phi) = 0$). In particular, we include as a first study the minimal coupling $\xi = 0$ since the fields equations do not allow to consider a potential in this case.

For a scalar field minimally coupled to gravity (i.e., $\xi = 0$), the Einstein equations become of first order for the scalar field. In particular, combining Eqs. (6) and (8) we obtain

$$T_{uv} + T_{yy} = (\partial_y\Phi)^2 = 0, \tag{11}$$

$$T_{uv} - T_{yy} = 2U(\Phi) = 0, \tag{12}$$

and conclude that the self-interaction must be absent while the scalar field depends only on the retarded time, $\Phi = \Phi(u)$. The remaining equation (9) now reduces to

$$\frac{1}{2}\partial_{yy}^2F = \kappa\left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2, \tag{13}$$

and integrates as

$$F(u, y) = \kappa\left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2 y^2 + F_1(u)y + F_0(u), \tag{14}$$

where F_1 and F_0 are two integration functions. It is well-known that any dependence up to first grade on the front-wave coordinate y in F can be eliminated through a coordinate transformation [3]. For example, for a generic dependence

$$F(u, y) = F_2(u)y^2 + F_1(u)y + F_0(u), \tag{15}$$

the corresponding transformation which permits to eliminate F_1 and F_0 is

$$(u, v, y) \mapsto \left(u, v - \tfrac{1}{2}\tfrac{dB}{du}(2y - B) + \tfrac{1}{4}\int du(F_1 B + 2F_0), y - B\right), \tag{16}$$

where $B = B(u)$ is a function satisfying the linear equation

$$\frac{d^2 B}{du^2} + F_2 B = -\frac{1}{2}F_1.$$

Applying the coordinate transformation (16) to our case we obtain the following solution

$$ds^2 = -\kappa \left(\frac{d\Phi}{du}\right)^2 y^2 du^2 - 2dudv + dy^2, \tag{17a}$$

$$\Phi = \Phi(u), \tag{17b}$$

corresponding to the geometry of a plane wave with profile fixed by the scalar field which depends arbitrarily on the retarded time.

A natural question following from this analysis is to ask whether such free scalar field configurations still exist if one includes a nonminimal coupling to gravity. The procedure is similar to what we done for the minimal case. Indeed, the absence of potential together with Eq. (8) imply

$$\partial_y \Phi = 0,$$

from which we conclude again that the scalar field depends only on the retarded time and, hence, Eqs. (6) and (7) are trivially satisfied. The remaining independent Einstein equation (9) becomes now

$$\frac{1}{2}\partial_{yy}^2 F = \frac{\kappa}{1 - \kappa \xi \Phi^2}\left[(1 - 2\xi)\left(\frac{d\Phi}{du}\right)^2 - 2\xi\Phi\frac{d^2\Phi}{du^2}\right], \tag{18}$$

and the absence of dependence on y in the right hand side gives straightforwardly

$$F(u, y) = \frac{\kappa}{1 - \kappa \xi \Phi^2}\left[(1 - 2\xi)\left(\frac{d\Phi}{du}\right)^2 - 2\xi\Phi\frac{d^2\Phi}{du^2}\right]y^2 + F_1(u)y + F_0(u). \tag{19}$$

As done previously, the coordinate transformation (16) allows to eliminate the functions F_1 and F_0. Thus, the solution in the free case corresponds to a plane wave with its profile determined again from the wave profile of the scalar field

$$ds^2 = -\frac{\kappa}{1 - \kappa \xi \Phi^2}\left[(1 - 2\xi)\left(\frac{d\Phi}{du}\right)^2 - 2\xi\Phi\frac{d^2\Phi}{du^2}\right]y^2 du^2 - 2dudv + dy^2, \tag{20a}$$

$$\Phi = \Phi(u). \tag{20b}$$

It is interesting to note that the value $\xi = 0$ which corresponds to the minimal coupling is not singular in the expression (20) and yields precisely to the minimal solution (17).

We conclude that in absence of potential, the system of equations (2-3) supports plane wave gravitational fields for which the source also behaves like a wave. In the next section, we show that the introduction of a potential whose form is dictated by the field equations breaks the wavy behavior of the scalar field while the geometries are now interpreted as *pp* waves.

IV. SELF-INTERACTING SCALAR FIELDS: *PP* WAVES

We now consider a scalar field nonminimally coupled to gravity with a self-interaction potential. In order to achieve this analysis, it is judicious to make the following redefinition

$$\Phi = \frac{1}{\sigma^{2\xi/(1-4\xi)}}, \tag{21}$$

which obviously excludes the value $\xi = 1/4$. This case deserves a separate analysis since Eqs. (6) and (7) integrate as logarithms for this specific value (see Appendix A). The equations (6-7) expressed in terms of σ become

$$\partial^2_{yy}\sigma = 0, \tag{22a}$$
$$\partial^2_{uy}\sigma = 0, \tag{22b}$$

which implies that σ is separable in u and y and linear in y, i.e.,

$$\sigma(u,y) = 2\sqrt{\lambda}y + f(u), \tag{23}$$

where λ is a positive constant and f is an undetermined function of the retarded time. Inserting the above expression into Eq. (8) imposes the self-interaction potential to be of the form

$$U_\xi(\Phi) = \frac{8\xi^2\lambda}{(1-4\xi)^2}\Phi^{(1-2\xi)/\xi}. \tag{24}$$

We would like to stress that the emergence of such potential is interesting. Indeed, as it is well-known, a scalar field conformally coupled to $D-$dimensional gravity requires the non-minimal coupling parameter to be chosen as $\xi = \xi_D = (D-2)/[4(D-1)]$. Surprisingly, the allowed self-interaction potential which does not spoil the conformal invariance of the scalar field is precisely the one obtained here (24) when it is written in terms of the corresponding conformal coupling. For example in three dimensions, the conformal coupling is $\xi = 1/8$ and so the potential (24) becomes

$$U_{1/8}(\Phi) = \frac{\lambda}{2}\Phi^6, \tag{25}$$

which corresponds to the conformally invariant potential in three dimensions. Another interesting value is $\xi = 1/2$ for which the potential (24) becomes a positive constant. Hence, taking the value $\xi = 1/2$ is equivalent to consider the Einstein equations with an effective positive cosmological constant of value

$$\Lambda = \kappa U_{1/2}(\Phi) = 2\kappa\lambda, \tag{26}$$

and without self-interaction potential.

We now go back to our analysis for which we have fully determined the scalar source. As a simple check, it can be shown that the scalar field given by Eqs. (21) and (23) is a solution of the nonlinear Klein-Gordon equation on a *pp* wave background (10) for the potential (24). Hence, it remains to obtain the metric function F from Eq. (9). In order to achieve this task, it is convenient to define new independent variable and function as

$$x = \kappa\xi(2\sqrt{\lambda}y + f)^{-4\xi/(1-4\xi)}, \tag{27a}$$
$$H(u,x) = \frac{F}{2\sqrt{\lambda}y + f} - \frac{1}{2\lambda}\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}. \tag{27b}$$

With the above redefinitions, Eq. (9) is now written as

$$x(x-1)\partial_{xx}^2 H + \frac{(12\xi - 1)x + 1 - 8\xi}{4\xi}\partial_x H - \frac{1 - 4\xi}{4\xi}H = 0, \tag{28}$$

from which we recognize the hypergeometric differential equation [4]. The general solution of equation (28) is

$$H(u,x) = F_1(u)\, {}_2\tilde{F}_1\left(1, \frac{4\xi - 1}{4\xi}; \frac{8\xi - 1}{4\xi}; x\right) + F_2(u)\left(\frac{x}{\kappa\xi}\right)^{(1-4\xi)/(4\xi)}, \tag{29}$$

where F_1 and F_2 are integration functions, and ${}_2\tilde{F}_1(a, b; c; x)$ denotes the hypergeometric function with parameters a, b, and c. Obviously, the above representation in terms of the hypergeometric function is only valid when the hypergeometric function is well-defined. As it is shown in details in Appendix B, the nonminimal coupling values $\xi_n = 1/[4(2 + n)]$, $n = 0, 1, 2, \ldots$, are excluded for this reason. Their corresponding pp wave configurations will be analyzed separately in this Appendix. For the other values we evaluate the solution (29) in the original variables (27) and we perform the following coordinate change, which allows to eliminate from the metric up to the first grade dependence on the front-wave coordinate y,

$$(u, v, y) \mapsto \left(u, v + \tfrac{1}{4\lambda}\left(\mathrm{d}f/\mathrm{d}u\right)\left(2\sqrt{\lambda}y + f\right) - \tfrac{1}{8\lambda}\int \mathrm{d}u\left[\left(\mathrm{d}f/\mathrm{d}u\right)^2 - 4\lambda F_2\right], y + \tfrac{f}{2\sqrt{\lambda}}\right). \tag{30}$$

This transformation is equivalent to put $f = 0$ and $F_2 = 0$ and hence, it clearly shows that the undetermined dependence on the retarded time of the scalar field (23) can be removed. This fact is an obvious consequence of the self-interacting character of the source. Finally, for a generic nonminimal coupling parameter the solution is given by

$$ds^2 = -F_1(u)\, {}_2\tilde{F}_1\left(1, \frac{4\xi - 1}{4\xi}; \frac{8\xi - 1}{4\xi}; \kappa\xi\Phi^2\right)2\sqrt{\lambda}y du^2 - 2dudv + dy^2, \tag{31a}$$

$$\Phi = (2\sqrt{\lambda}y)^{-2\xi/(1-4\xi)}. \tag{31b}$$

More precisely, for $\xi \neq 1/2$, the previous solution is interpreted as a pp wave for a self-interacting scalar field nonminimally coupled to gravity. For the value $\xi = 1/2$, the solution (31) is well defined and can be expressed in terms of the effective cosmological constant (26) as

$$ds^2 = -F_1(u)\mathrm{arctanh}(\sqrt{\Lambda}y)du^2 - 2dudv + dy^2, \tag{32a}$$

$$\Phi = \sqrt{\frac{2\Lambda}{\kappa}}y, \tag{32b}$$

and can be seen as a solution for a free scalar field nonminimally coupled to gravity (with parameter $\xi = 1/2$) in presence of a positive cosmological constant. It is interesting to note that in contrast with the free solutions of Sec. III, where the cosmological constant is absent, its introduction breaks the wave behavior of the scalar field, inducing a linear dependence on front coordinate and, changes the plane wave character of the gravitational field.

This last remarks opens naturally the discussion about the introduction of a cosmological constant in our original system of equations. It is simple to see that in the free case, the only

possibility occurs for the nonminimal coupling parameter value $\xi = 1/2$ and for a positive cosmological constant. The solution is precisely the one obtained in Eqs. (32). In the self-interacting case the introduction of a cosmological constant is trivial. It is equivalent of having an effective potential

$$U_{\text{eff}}(\Phi) = U(\Phi) + \frac{\Lambda}{\kappa},$$

without cosmological constant, and hence the results of this section applies for this effective potential. Returning to the original potential this is equivalent to add the cosmological term to both sides of the Einstein equations.

This analysis completes the study of scalar fields (including or not a self-interaction) nonminimally coupled to pp waves, when gravitation is described by the standard Einstein tensor. In the next section we address the same problem when gravitation is governed by topologically massive gravity for which the Einstein tensor is supplemented by the Cotton tensor. For simplicity, this analysis is done only in the free case.

V. TOPOLOGICALLY MASSIVE GRAVITY *PP* WAVES

In this section we extend the scope of this work to topologically massive gravity which is an alternative gravitational theory in three dimensions introduced by Deser, Jackiw, and Templeton [5]. This theory is obtained by adding to the $2 + 1$ gravity action (1) the topological Chern-Simons term for the local Lorentz group. The corresponding field equations obtained by varying the metric read

$$\frac{1}{\mu}C_{\alpha\beta} + G_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{33}$$

where μ is the topological mass, $C_{\alpha\beta}$ is the symmetric, conserved, and traceless Cotton tensor defined by

$$C^{\alpha\beta} = \frac{1}{\sqrt{-g}}\epsilon^{\alpha\sigma\nu}D_{\sigma}\left(R_{\nu}{}^{\beta} - \frac{1}{4}\delta_{\nu}{}^{\beta}R\right), \tag{34}$$

while the energy-momentum is given by Eq. (4). Note that for lightlike sources, the pp wave geometries of this theory have been previously considered in Refs. [6]

For later convenience, we first consider the vacuum case of equations (33) for a pp wave background. As it is well-known, one of the most important differences between standard $2+1$ gravity and topologically massive gravity is that, this later supports gravitational waves in vacuum since it has at least one propagating degree of freedom. For the pp wave geometry, the only nonvanishing component of the Cotton tensor is $C_{uu} = \frac{1}{2}\partial_{yyy}^3 F$, and consequently in absence of sources, the only nonvanishing equation is

$$\frac{1}{\mu}C_{uu} + G_{uu} = \frac{1}{2\mu}\partial_{yyy}^3 F + \frac{1}{2}\partial_{yy}^2 F = 0. \tag{35}$$

This equation is integrated as

$$F(u, y) = F_2(u)\mathrm{e}^{-\mu y} + F_1(u)y + F_0(u), \tag{36}$$

and, after eliminating up to the first grade dependence on y, gives the following pp wave geometry

$$ds^2 = -F_2(u)\mathrm{e}^{-\mu y}du^2 - 2dudv + dy^2. \tag{37}$$

It is easy to see that the metric function F satisfies the linear Klein-Gordon equation

$$\Box F = \mu^2 F, \tag{38}$$

with mass μ which justifies the massive character of the theory.

We now consider the same problem with the scalar source. Since the Cotton tensor involves third-order derivatives of the metric function F, we restrict ourselves to the free case (i.e., without a self-interacting potential). As seen previously, the Cotton tensor has only a contribution along the component uu of the equations (33) and hence the analysis done in Sec. III for the free case is still valid for the other components of the equations. This means that the scalar field depends only on the retarded time $\Phi = \Phi(u)$, while the uu equation becomes now

$$\frac{1}{2\mu}\partial^3_{yyy}F + \frac{1}{2}(1 - \kappa\xi\Phi^2)\partial^2_{yy}F = \kappa\left[(1 - 2\xi)\left(\frac{d\Phi}{du}\right)^2 - 2\xi\Phi\frac{d^2\Phi}{du^2}\right]. \tag{39}$$

The integration of this equation yields to the following solution

$$F(u, y) = F_2(u)\exp\left[-\mu(1 - \kappa\xi\Phi^2)y\right] + \frac{\kappa}{1 - \kappa\xi\Phi^2}\left[(1 - 2\xi)\left(\frac{d\Phi}{du}\right)^2 - 2\xi\Phi\frac{d^2\Phi}{du^2}\right]y^2$$
$$+ F_1(u)y + F_0(u), \tag{40}$$

from which the functions F_0 and F_1 can be eliminated, and we end with a configuration given by

$$ds^2 = -\left\{F_2(u)\exp\left[-\mu(1 - \kappa\xi\Phi^2)y\right] + \frac{\kappa}{1 - \kappa\xi\Phi^2}\left[(1 - 2\xi)\left(\frac{d\Phi}{du}\right)^2 - 2\xi\Phi\frac{d^2\Phi}{du^2}\right]y^2\right\}du^2$$
$$- 2dudv + dy^2, \tag{41a}$$
$$\Phi = \Phi(u). \tag{41b}$$

As a first constatation, the inclusion of the topological mass has broken the plane wave behavior of the gravitational field present in $2 + 1$ gravity for a free field (20). In fact, the term $\mu_{\text{eff}} = \mu(1 - \kappa\xi\Phi^2)$ acts as an effective topological mass since the corresponding metric function F satisfies

$$\Box F = \mu_{\text{eff}}{}^2 F + \ldots, \tag{42}$$

i.e., the Klein-Gordon equation with a retarded-time-dependent mass term plus corrective terms due to the presence of the scalar source. The above effective topological mass reduces to the vacuum value $\mu_{\text{eff}} = \mu$ for minimal coupling $\xi = 0$. Indeed, in the minimal case the gravitational field becomes

$$ds^2 = -\left[F_2(u)e^{-\mu y} + \kappa\left(\frac{d\Phi}{du}\right)^2 y^2\right]du^2 - 2dudv + dy^2, \tag{43}$$

and consists in a simple superposition of the vacuum pp waves of topologically massive gravity (37) and the $2 + 1$ gravity planes waves (17) with a free minimally coupled scalar field as source. This is related to the fact that the equation (39) is a inhomogeneous linear

differential equation whose solution is the sum of a particular solution, determined from the scalar inhomogeneity, with a solution of the corresponding homogenous system. Obviously, in absence of a scalar field ($\Phi = 0$) we recover from the above solution or from the expression (41), in presence of nonminimal coupling, the vacuum massive pp waves (37).

For an huge value of the topological mass ($\mu \gg 1$), the Einstein tensor is dominant and one would expect that at this limit the solutions (41) become the plane waves (20) of $2 + 1$ gravity. This is indeed the case provided the quantity $(1 - \kappa\xi\Phi^2)$ to be positive. This assumption is not too restrictive since in presence of nonminimal coupling to gravity the term $\kappa_{\text{eff}} = \kappa/(1 - \kappa\xi\Phi^2)$ acts as an effective gravitational constant, and it must be positive at least in the weak gravity limit in order to recover the intuitive attractive behavior of gravity. On the light of this, under this assumption the solution (41) can also be interpreted as a superposition of the vacuum pp waves of topologically massive gravity (37) taking as topological mass the effective one, $\mu_{\text{eff}} = \mu(1 - \kappa\xi\Phi^2)$, together with the $2 + 1$ gravity plane waves (20) which have a free nonminimally coupled scalar field as source.

In the next section, we shall see that the solutions described here at a special limit when the topological mass goes to zero describe new pp wave solutions of conformal gravity.

VI.   CONFORMAL GRAVITY PP WAVES

A.   Free conformal gravity pp waves from topologically massive gravity

At the limit of small topological mass $\mu \to 0$ with $\mu\kappa \sim 1$, the topologically massive gravity equations (33) become [6]

$$C_{\alpha\beta} = \tilde{\kappa}T_{\alpha\beta}, \tag{44}$$

where $\tilde{\kappa}$ is a dimensionless constant. Thus, at this special limit, the contribution of the Einstein tensor disappears and gravity is only governed by the Cotton tensor. At the first sight, since the Cotton tensor is conformally invariant, this equation (44) seems to restrict the source to be also conformally invariant. Two recent works have been done in this direction where pp waves configurations have been obtained for a free conformal scalar field [1] and later extended by the inclusion of a potential that does not spoil the conformal invariance [2]. In both cases, this corresponds to choose the conformal coupling parameter $\xi = 1/8$. However, it appears that the solutions derived here for topologically massive gravity (41) at the limit discussed above generate new solutions of conformal gravity equations (44) for any arbitrary nonvanishing value of the nonminimal coupling parameter ξ. Indeed, at the limit $\mu \to 0$ with $\mu\kappa \sim 1$, the solutions (41) become

$$ds^2 = -\left[F_2(u)e^{\tilde{\kappa}\xi\Phi^2 y} + 2\Phi^{\frac{1-4\xi}{2\xi}}\frac{\mathrm{d}}{\mathrm{d}u}\left(\Phi^{\frac{2\xi-1}{2\xi}}\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)y^2\right]du^2 - 2dudv + dy^2, \tag{45a}$$

$$\Phi = \Phi(u), \tag{45b}$$

and, a simple check shows that these limiting configurations are effectively solutions of conformal gravity (44). In particular, for the conformal coupling $\xi = 1/8$, we recover exactly the pp wave solutions of Ref. [1]. For $\xi \neq 1/8$, the matter source is not *a priori* conformally invariant and so a natural question is to ask why these limiting configurations are solutions of conformal gravity. The "miraculous" lies in the fact that pp wave backgrounds impose

to the matter to have only one nonvanishing component of the energy-momentum tensor, namely T_{uu}, and since $g^{uu} = 0$, the energy-momentum tensor is traceless on-shell. In other words, the conformal character of the equations (44) is preserved on-shell. Various questions emerge from the last observation. For example, one can asks whether the limiting solutions (45) are the only ones of conformal gravity with a free source for any value of the nonminimal coupling parameter or if their exist other configurations. A simple calculation shows that for $\xi \neq 0$ the limiting configurations (45) are the only pp wave solutions of conformal gravity equations (44). Since the minimal coupling limit is singular in Eq. (45), we solve independently the original equations (44) in the case of the minimal coupling $\xi = 0$ and we get the following solution

$$ds^2 = -\left(\frac{\tilde{\kappa}}{3} \left(\frac{d\Phi}{du} \right)^2 y^3 + F_2(u)y^2 \right) du^2 - 2dudv + dy^2, \tag{46a}$$

$$\Phi = \Phi(u). \tag{46b}$$

Note that in absence of source (i.e., $\Phi = 0$) one recovers the plane wave solution of the vacuum conformal gravity.

The argument which consists of looking for configurations that have a traceless energy-momentum tensor on-shell can be applied for any matter source (not necessarily a free scalar field). In particular, in what follows, we consider the self-interacting case for an arbitrary value of the nonminimal coupling parameter.

B. Self-interacting conformal gravity pp waves

Motivated by the previous study, we explore the existence of pp wave configurations for conformal gravity with a self-interacting source and for a generic value of the nonminimal coupling parameter. Due to the fact that for a pp wave ansatz, C_{uu} is the only nonvanishing component of the Cotton tensor, the arguments applied in Sec. IV are still valid. In particular, the potential must have the form given by the expression (24) while the scalar field is expressed by Eqs. (21) and (23). Making the following redefinitions of the involved variables

$$x = \frac{\tilde{\kappa}\xi(1 - 4\xi)}{2\sqrt{\lambda}(1 - 8\xi)} \left(2\sqrt{\lambda}y + f \right)^{(1-8\xi)/(1-4\xi)}, \tag{47a}$$

$$H(u, x) = \frac{F}{2\sqrt{\lambda}y + f} - \frac{1}{2\lambda} \frac{d^2 f}{du^2}, \tag{47b}$$

the component uu of the equations (44) can be written as

$$x^2 \partial^3_{xxx} H - x(x - 3)\partial^2_{xx} H - 2\frac{(1 - 8\xi)^2 x + 4\xi(1 - 6\xi)}{(1 - 8\xi)^2} \partial_x H - \frac{4\xi(1 - 4\xi)}{(1 - 8\xi)^2} H = 0. \tag{48}$$

The above equation is the generalized hypergeometric differential equation [4] for which the general solution reads

$$
\begin{aligned}
H(u,x) \;=\; & F_1(u)\, {}_1\tilde{F}_1\left(\frac{1-4\xi}{1-8\xi}; \frac{2(1-6\xi)}{1-8\xi}; x\right) + F_2(u)\left(\frac{2\sqrt{\lambda}(1-8\xi)}{\tilde{\kappa}\xi(1-4\xi)}x\right)^{-\frac{1-4\xi}{1-8\xi}} \\
& + F_3(u)\left(\frac{2\sqrt{\lambda}(1-8\xi)}{\tilde{\kappa}\xi(1-4\xi)}x\right)^{\frac{1-4\xi}{1-8\xi}} {}_2\tilde{F}_2\left(1, \frac{2(1-4\xi)}{1-8\xi}; \frac{3-16\xi}{1-8\xi}, \frac{2(1-6\xi)}{1-8\xi}; x\right).
\end{aligned}
\tag{49}
$$

Here ${}_1\tilde{F}_1(a; b; x)$ and ${}_2\tilde{F}_2(a, b; c, d; x)$ denote the corresponding generalized hypergeometric functions [4]. Returning to the original variables by means of Eqs. (47) and after making the coordinate change (30) we obtain the final solution

$$
\begin{aligned}
ds^2 \;=\; & -\left[F_1(u)\, {}_1\tilde{F}_1\left(\frac{1-4\xi}{1-8\xi}; \frac{2(1-6\xi)}{1-8\xi}; \frac{\tilde{\kappa}\xi(1-4\xi)}{2\sqrt{\lambda}(1-8\xi)}\Phi^{\frac{8\xi-1}{2\xi}}\right) 2\sqrt{\lambda}\,y \right. \\
& \left. + F_3(u)\, {}_2\tilde{F}_2\left(1, \frac{2(1-4\xi)}{1-8\xi}; \frac{3-16\xi}{1-8\xi}, \frac{2(1-6\xi)}{1-8\xi}; \frac{\tilde{\kappa}\xi(1-4\xi)}{2\sqrt{\lambda}(1-8\xi)}\Phi^{\frac{8\xi-1}{2\xi}}\right) 4\lambda y^2 \right] du^2 \\
& - 2\,du\,dv + dy^2, \\
\Phi \;=\; & (2\sqrt{\lambda}\,y)^{-2\xi/(1-4\xi)}.
\end{aligned}
$$

$$\tag{50a}$$
$$\tag{50b}$$

We shall not intent here to cover the singular values of the nonminimal coupling parameter in the above solution. This can be done along the same line than in the self-interacting $2+1$ gravity case of Appendix B. We just notice that the conformal coupling $\xi = 1/8$ belongs to those singular values but this case has already been derived in Ref. [2].

VII. CONCLUSIONS

As it is well-known, pure gravitational waves are forbidden in three dimensions. A natural way to circumvent this problem is to consider a matter source as it has been done here for a scalar field nonminimally coupled to a *pp* wave with or without self-interaction potential. For Einstein gravity, in the free case, we obtain gravitational plane wave solutions for which the scalar field also behaves like a wave. Additionally, the gravitational wave profile is fixed in terms of the scalar one.

The introduction of a self-interaction potential has several consequences. Firstly, its form is dictated by the field equations which only allow power-law potentials with powers given in terms of the nonminimal coupling parameter. For the conformal coupling, this potential reduces exactly to the conformally invariant one in three dimensions. Secondly, the presence of the self-interaction breaks explicitly the wavy behavior of the scalar field that was present in the free case. Indeed, the scalar field loses its arbitrary dependence on the retarded time, and is now an explicit function of the wave-front coordinate. For the special value of the coupling parameter $\xi = 1/2$, the corresponding potential becomes a positive constant, and hence the related configuration can be seen as a free scalar field that solve the Einstein equations in presence of an effective positive cosmological constant. The special value $\xi = 1/4$, studied separately in Appendix A, is also interpreted in a different way. In this case the potential reduces to a mass term, and consequently the system describes a free massive scalar field with the corresponding nonminimal coupling to gravity.

We have also considered the natural extension for which pp waves are rigged by topologically massive gravity with a free nonminimally coupled scalar field acting as a source. In the particular case of minimal coupling, the solutions can be seen as a simple superposition of the vacuum pp waves of topologically massive gravity and the Einstein gravity plane waves with a free scalar source. In the nonminimal case, the solutions can be interpreted again as a superposition of the corresponding solutions of topologically massive gravity and Einstein gravity, provided the topological mass is changed by an effective one depending on the nonminimal coupling.

At the small topological mass limit with a huge gravitational constant, new solutions of conformal gravity with a free source have been obtained for any arbitrary value of the nonminimal coupling parameter. This is not in contradiction with the conformal character of conformal gravity since these limiting solutions have a traceless energy-momentum tensor on-shell. This fact is due to the particular pp wave ansatz that restricts the matter source to have only one nonvanishing energy-momentum tensor component along the retarded time. Motivated by the results of the above limit, we have also extended the previous nonminimally coupled configurations to the self-interacting case. Interestingly, the allowed potentials are exactly the same than those arising in the case of Einstein gravity.

It would be interesting to explore the existence of other background geometries that allow special superposition of solutions as those arising here. From this study, it is also natural to consider other matter source to conformal gravity that preserves the conformal invariance on-shell. Another interesting work will consist to extend these considerations in arbitrary D dimension and to see whether these features are still valid or were only specific to the three-dimensional case. In this case, an interesting option would be to consider front wave geometries which are not necessarily planes.

APPENDIX A: PP WAVES FROM NONMINIMAL COUPLING $\xi = 1/4$

For the specific value $\xi = 1/4$ of the nonminimal coupling parameter, our substitution (21) for a self-interacting scalar field is not valid and instead we consider

$$\Phi = \frac{1}{\sqrt{\kappa}} e^{\sigma}. \tag{A1}$$

With this substitution the function σ satisfy the same equations than in the generic case, i.e., Eqs. (22), and is given by

$$\sigma(u, y) = my + f(u), \tag{A2}$$

while the allowed potential is now a simple mass term

$$U_{1/4}(\Phi) = \frac{1}{2}m^2\Phi^2. \tag{A3}$$

The remaining Einstein equation (9) is expressed as

$$\left(4e^{-2(my+f)} - 1\right)\partial_{yy}^2 F - 2m\partial_y F + 4\frac{d^2 f}{du^2} = 0, \tag{A4}$$

whose solution reads

$$F(u,y) = \left(F_1(u) + \frac{1}{m^2}\frac{d^2 f}{du^2}\right)\ln\left(4e^{-2(my+f)} - 1\right) + \frac{2}{m^2}\frac{d^2 f}{du^2}(my+f) + F_2(u). \tag{A5}$$

After performing the coordinate transformation (16), we finally obtain the following solution

$$ds^2 = -F_1(u)\ln\left(4e^{-2my} - 1\right)du^2 - 2dudv + dy^2, \tag{A6}$$

$$\Phi = \frac{1}{\sqrt{\kappa}}e^{my}, \tag{A7}$$

which describe a free massive scalar field nonminimally coupled to a pp wave with parameter $\xi = 1/4$.

APPENDIX B: PP WAVES FROM NONMINIMAL COUPLINGS $\xi_n = 1/[4(2+n)]$

Here we analyze the singular cases not covered within the self-interacting solution (29) for Einstein gravity. The hypergeometric function $_2\tilde{F}_1(a,b;c;x)$ is not defined when the parameter c is equal to a non-positive integer $-m$ provided that a or b is not equal to a negative integer $-n$ with $n < m$. In our case, solution (29) is of the form $_2\tilde{F}_1(1,b;b+1;x)$ where $b = (4\xi - 1)/(4\xi)$ and consequently, we must exclude from the previous analysis the following values of the nonminimal coupling parameter

$$\xi_n = \frac{1}{4(n+2)}, \qquad \text{where} \quad n \in \mathbb{N}, \tag{B1}$$

which includes in particular the conformal coupling $\xi_0 = 1/8$. The self-interaction potentials corresponding to these special nonminimal couplings are

$$U_n(\Phi) = \frac{\lambda}{2(n+1)^2}\Phi^{2(2n+3)}. \tag{B2}$$

We now analyze the pp waves configurations for these particular values. For $\xi = \xi_n$, the integration of equation (28) for each n leads to the following solution

$$H_n(x) = \left\{F_1(u)\left[\ln\left(1 - \frac{1}{x}\right) + \sum_{l=1}^{n+1}\frac{1}{lx^l}\right] + F_2(u)\right\}\left(\frac{x}{\kappa\xi_n}\right)^{n+1}. \tag{B3}$$

Recovering the original variables from definitions (27) and using again the coordinate transformation (30) the final solution for the discrete values of the nonminimal coupling parameter $\xi_n = 1/[4(2+n)]$, $n = 0, 1, 2, \ldots$, reads

$$ds^2 = -F_1(u)\left[\ln\left(1 - \frac{1}{\kappa\xi_n\Phi^2}\right) + \sum_{l=1}^{n+1}\frac{1}{l(\kappa\xi_n\Phi^2)^l}\right] du^2 - 2dudv + dy^2, \qquad \text{(B4a)}$$

$$\Phi = \frac{1}{(2\sqrt{\lambda}y)^{1/[2(n+1)]}}. \qquad \text{(B4b)}$$

In particular, when the source is conformally invariant [i.e., $\xi_0 = 1/8$ with the conformal potential (25)] the above expression reduces to

$$ds^2 = -F_1(u)\left[\ln\left(1 - \frac{16\sqrt{\lambda}y}{\kappa}\right) + \frac{16\sqrt{\lambda}y}{\kappa}\right] du^2 - 2dudv + dy^2, \qquad \text{(B5a)}$$

$$\Phi = \frac{1}{\sqrt{2\sqrt{\lambda}y}}. \qquad \text{(B5b)}$$

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