Higher-dimensional black holes with a conformally invariant Maxwell source

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We consider an action for an abelian gauge field for which the density is given by a power of the Maxwell Lagrangian. In d spacetime dimensions this action is shown to enjoy the conformal invariance if the power is chosen as $d/4$. We take advantage of this conformal invariance to derive black hole solutions electrically charged with a purely radial electric field. Because of considering power of the Maxwell density, the black hole solutions exist only for dimensions which are multiples of four. The expression of the electric field does not depend on the dimension and corresponds to the four-dimensional Reissner-Nordström field. Using the Hamiltonian action we identify the mass and the electric charge of these black hole solutions.

I. INTRODUCTION

The fundamental paradigm of General Relativity is the non trivial interaction between matter and geometry which is mathematically encoded through the Einstein equations. These later relate the geometry of the spacetime with the matter source which depends explicitly on the metric, and hence the complexity of the Einstein equations is considerably increased. In general, the Einstein equations with a matter source possessing the conformal invariance can be simplified. Indeed, in absence of the cosmological constant, a traceless energy-momentum tensor implies that the scalar curvature is zero restricting the possible spacetimes. A well-known example is given by the so-called BBMB black hole in four dimensions for which the matter is described by a scalar field nonminimally coupled to gravity with the conformal coupling (and also with an electric field) [1, 2]. In this example, the conformal character of the matter source has been crucial since the solution has been derived using the machinery of conformal transformations applied to minimally coupled scalar fields [2]. Unfortunately, the hope that the conformal symmetry of the scalar field matter source was behind the existence of black hole solutions in the case of static spherically symmetric spacetimes has been ruined by Xanthopoulos & Dialynas [3] and Klimčík [4]. They have shown that in higher dimensions, a scalar field conformally coupled to gravity do not exhibit black hole solutions.

Conformal symmetry of the matter source can also be useful for gravity in presence of a cosmological constant. In this case, the traceless character of the source imposes the spacetime to be of constant scalar curvature. However, in this case there does not exist a no-hair theorem that rules out regular black hole solutions on and out of the event horizon. In fact, black hole solutions with nonvanishing cosmological constant have been obtained in the case of a conformally and self-interacting coupled scalar field in three dimensions [5, 6] and in four dimensions [7, 8].

The first black hole solution derived for which the matter source is conformally invariant is the Reissner-Nordström solution in four dimensions. Indeed, in this case the source is given by the Maxwell action which enjoys the conformal invariance in four dimensions. The Reissner-Nordström is an electrically charged but non-rotating black hole solution and, is the unique spherically symmetric and asymptotically flat solution of the Einstein-Maxwell equations. Later, this solution has been extended in higher dimensions where the Maxwell action does not possess the conformal symmetry [9].

A legitimate question to ask is whether there exists an extension of the Maxwell action in arbitrary dimension that possesses the conformal invariance. The answer is positive and the conformally invariant Maxwell action is given by

$$I_\mathcal{M} = -\alpha \int d^d x \, \sqrt{-g} \, (F_{\mu\nu} F^{\mu\nu})^{\frac{d}{4}}, \qquad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell tensor, and α is a constant. It is simple to see that under a conformal transformation which acts on the fields as $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $A_\mu \to A_\mu$, the action (1) remains unchanged [10]. Note that in four dimensions, the conformal action (1) reduces to the Maxwell action as it should be. The energy-momentum tensor associated to $I_\mathcal{M}$ is given by

$$T_{\mu\nu} = 4\alpha \left(\frac{d}{4} F_{\mu\rho} F_\nu{}^\rho \, F^{\frac{d}{4}-1} - \frac{1}{4} g_{\mu\nu} F^{\frac{d}{4}} \right) \qquad (2)$$

where $F = F_{\alpha\beta} F^{\alpha\beta}$ is the Maxwell invariant, and the conformal invariance of the action is encoded by the traceless condition $T_\mu^\mu = 0$. Note that there exists another conformally invariant extension of the Maxwell action in higher dimensions for which the Maxwell field is

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replaced by a $d/2-$form with d even [11]. The black hole solutions of this theory were discussed in [12].

In what follows, we are going to consider the action (1) as the matter source of the Einstein equations. The idea is to take advantage of the conformal symmetry to construct the analogues of the four-dimensional Reissner-Nordström black hole solutions in higher dimensions. The Ansatz for the Maxwell tensor is restricted to be given by a non zero electric field radial F_{tr}. Because of the form of the energy-momentum tensor (2), the Ansatz on the Maxwell tensor automatically restricts the possible dimensions to be only multiples of four. For these dimensions, we derive the most general black hole solutions in the static and spherically symmetric case. The metric solution depends on two constants which are related to the mass and to the electric charge of the black hole. Moreover, as shown below, the extended version of the Maxwell equations imposes the electric field to have the same expression independently of the dimension. Finally, from a Hamiltonian action we derive the mass and the electric charge for these solutions, expressed as surface integrals.

II.   CHARGED BLACK HOLES SOLUTIONS

In dimension $d = 4 + 4p$ with $p \in \mathbb{N}$, we consider the Einstein action with an extended Maxwell action given by

$$I[g_{\mu\nu}, A_\mu] = \int d^{4+4p}x \sqrt{-g}\Big[\frac{R}{2\kappa} - \alpha\,(F_{\mu\nu}F^{\mu\nu})^{p+1}\Big], \quad (3)$$

where R is the scalar curvature and κ is the gravitational constant. The field equations obtained by varying the metric and the gauge field A_μ read respectively

$$G_{\mu\nu} = 4\kappa\alpha\Big((p+1)F_{\mu\rho}F_\nu{}^\rho\,F^p - \frac{1}{4}g_{\mu\nu}\,F^{p+1}\Big), \quad (4a)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\,F^p\right) = 0. \quad (4b)$$

We are looking for a static and spherically symmetric spacetime geometry whose line element is given by

$$ds^2 = -N^2(r)f^2(r)\,dt^2 + \frac{dr^2}{f^2(r)} + r^2\,d\Sigma_{4p+2}^2, \quad (5)$$

where $d\Sigma_{4p+2}^2 = \gamma_{ij}d\theta^i d\theta^j$ is the line element of the unit $(4p+2)-$ dimensional sphere, and we will use later on the notation $\gamma = \det(\gamma_{ij})$. As it was mentioned in the introduction, we are looking for a purely radial electric solution that means that the only non-vanishing component of the Maxwell tensor is given by F_{tr}. Because of the conformal invariance of the matter action, the scalar

curvature is zero and the Einstein equations are given by

$$R_t^t = -\frac{((Nf)'f)'}{N} - (4p+2)\frac{(Nf)'f}{rN} = (2p+1)U \quad (6a)$$

$$R_r^r = -\frac{((Nf)'f)'}{N} - (4p+2)\frac{f'f}{r} = (2p+1)U \quad (6b)$$

$$R_{\theta_i}^{\theta_i} = -\frac{(Nf)'f}{rN} - \frac{f'f}{r} + \frac{4p+1}{r^2}(1 - f^2) = -U \quad (6c)$$

for $i = 1, \cdots, 4p+2$, with $U = \kappa\alpha F^{p+1}$ and the prime denotes derivative with respect to the radial coordinate r. Subtracting Eqs. (6a) and (6b) we obtain that $N(r)$ is a constant, which can be set to 1 without loss of generality. Now, since the scalar curvature vanishes, $R = 0$, we can get from this equation the metric function $f^2(r)$, which is given by

$$f^2(r) = 1 - \frac{A}{r^{4p+1}} + \frac{B}{r^{4p+2}}, \quad (7)$$

where A is a constant proportional to the mass and B is a constant which is related to the electric charge as it will be shown later.

The extended Maxwell equation (4b) implies that the electric field F_{tr} is given by

$$F_{tr} = \frac{C}{r^2}, \quad (8)$$

where C is a constant. It is interesting to note that the expression of the electric field does not depend on the dimension and its value coincides with the Reissner-Nordström solution in four dimensions. The expression of the electric field (8) is compatible with the Einstein equations (4a) provided the constants B and C are related through

$$B = (-1)^p\,2^{p+1}\,C^{2p+2}\kappa\alpha. \quad (9)$$

Few remarks must be made to ensure that the metric describes a black hole. Firstly, one can choose the sign of the coupling constant α such that the energy density (the $T_{\hat{0}\hat{0}}$ component of the energy-momentum tensor in the orthonomal frame) is positive [1]. This means that $\text{sign}(\alpha) = (-1)^p$ and hence, the constant B is positive for all the value of p. In this case, in order to have real roots for $f^2(r)$, the constant A must be positive and the constant B must be chosen in the range

$$0 < B < (4p+1)\left(\frac{A}{4p+2}\right)^{\frac{4p+2}{4p+1}}. \quad (10)$$

Under these conditions, we have two roots: $r_- \in (0, b)$, and $r_+ \in (b, \infty)$, where

$$b = \left(\frac{A}{4p+2}\right)^{\frac{1}{4p+1}}.$$

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[1] We thank to anonymous referee for this comment

Finally, if A is positive and B is chosen as

$$B = (4p + 1)\left(\frac{A}{4p+2}\right)^{\frac{4p+2}{4p+1}},$$

we have a double root of $f^2(r)$ at

$$r_+ = \left(\frac{A}{4p+2}\right)^{\frac{1}{4p+1}}$$

producing an extreme black hole.

The black hole solutions discussed here have a single curvature singularity, which is located at $r = 0$. In what follows, we will see that the Hamiltonian action provides us a simple manner of identifying the mass and the electric charge for the previous black holes solutions.

A. Hamiltonian action

Here we are interested only in the static, spherically symmetric case without magnetic field, and hence it is enough to consider a *reduced* action principle. The class of metrics to be considered are given by (5) and the electromagnetic field has only an electric radial component given by P^r , which corresponds to the momentum conjugate to radial component of the gauge field A_r. The *reduced* action is decomposed in three pieces as

$$I^{\text{red}} = I^{\text{red}}_{\text{grav}} + I^{\text{red}}_{\mathcal{M}} + K, \tag{11}$$

where

$$\begin{aligned}
I^{\text{red}}_{\text{grav}} &= -(t_2 - t_1)A_{d-2}\frac{d-2}{2\kappa}\int dr N r^{d-2} \\
&\times \left[\frac{(f^2)'}{r} - \frac{d-3}{r^2}(1 - f^2)\right]
\end{aligned} \tag{12}$$

comes from the gravitational part of (3), while the reduced Hamiltonian version of the generalized Maxwell action (1) is given by

$$\begin{aligned}
I^{\text{red}}_{\mathcal{M}} &= (t_2 - t_1)A_{d-2}\int dr \left[\varphi \mathcal{P}'\right. \\
&\left. + \frac{(d-2)\alpha N(-2)^{\frac{d}{2(d-2)}}\mathcal{P}^{\frac{d}{d-2}}}{2(\alpha d)^{\frac{d}{d-2}}r^2}\right].
\end{aligned} \tag{13}$$

In the above expression, we have defined $\varphi \equiv A_t$ and

$$\mathcal{P} \equiv \gamma^{-1/2}P^r = \alpha d N F^{d/4-1}r^{d-2}F^{rt}$$

is the rescaled radial momentum. The symbol A_{d-2} is a short to denote the area of the $(d-2)$-dimensional unit sphere. The last term in (11), K, is a surface term that will be adjusted below.

Varying the reduced action (11) with respect to $N, f^2, \mathcal{P}$ and φ, the following equations are found

$$\frac{(f^2)'}{r} - \frac{d-3}{r^2}(1 - f^2) = \kappa\alpha\frac{(-2)^{\frac{d}{2(d-2)}}\mathcal{P}^{\frac{d}{d-2}}}{(\alpha d)^{\frac{d}{d-2}}r^d}, \tag{14}$$

$$N' = 0, \tag{15}$$

$$\varphi' = \frac{\alpha d N(-2)^{\frac{d}{2(d-2)}}\mathcal{P}^{\frac{2}{d-2}}}{2(\alpha d)^{\frac{d}{d-2}}r^2}, \tag{16}$$

$$\mathcal{P}' = 0, \tag{17}$$

whose general solutions in $d = 4(p+1)$ dimensions read

$$f^2 = 1 - \frac{A}{r^{4p+1}} + \frac{\alpha\kappa(-1)^p 2^{p+1}C^{2p+2}}{r^{4p+2}}, \tag{18}$$

$$N = N_\infty, \tag{19}$$

$$\varphi = \frac{N_\infty C}{r} + \varphi_\infty, \tag{20}$$

$$\mathcal{P} = \mathcal{P}_0, \tag{21}$$

where $C = (-1)^p(2^{-p}\mathcal{P}_0[\alpha(4p+4)]^{-1})^{1/(2p+1)}$. Note that Eq. (14) corresponds to the Hamiltonian constraint and Eq. (17) to the Gauss law. This general solution, which coincides exactly with the previous one obtained from the Einstein equations, has four integrations constants given by $A, \mathcal{P}_0, N_\infty, \varphi_\infty$. As we will show below, the values of N and φ at infinity, N_∞ and φ_∞, are conjugates to A and $\mathcal{P}_0$ respectively.

B. Mass and electric charge

The surface term K present in (11) is determined requiring the action has an extremum, i.e., $\delta I = 0$ within the class of fields considered here [13]. This implies that the variation of the boundary term is given by

$$\delta K = (t_2-t_1)A_{4p+2}\left((2p+1)\kappa^{-1}Nr^{4p+1}\delta f^2 - \varphi\delta\mathcal{P}\right) \tag{22}$$

for $r \to \infty$. Since $r^{4p+1}\delta f^2 = -\delta A + O(r^{-1})$ and $\delta\mathcal{P} = \delta\mathcal{P}_0$, we then get that

$$\delta K = (t_2 - t_1)A_{4p+2}\left(-(2p+1)\kappa^{-1}N_\infty\delta A - \varphi_\infty\delta\mathcal{P}_0\right). \tag{23}$$

The term K is the conserved charge associated to the "improper gauge transformations" produced by time evolution [14]. In our case, we have two transformations. The first one corresponds to time displacements for which the corresponding charge is the mass (M), and the second ones are the asymptotically constant gauge transformations of the electromagnetic field, where the electric charge (Q) is the corresponding charge. In term of the variation of the surface term, this is expressed as

$$\delta K = (t_2 - t_1)(-N_\infty\delta M - \varphi_\infty\delta Q). \tag{24}$$

where (N_∞, M) and (φ_∞, Q) are conjugate pairs. The comparison between (23) and (24) allows to identify the

conserved charges

$$\delta M = A_{4p+2}(2p+1)\kappa^{-1}\delta A, \qquad (25)$$
$$\delta Q = A_{4p+2}\delta \mathcal{P}_0. \qquad (26)$$

Finally, the integration of these variations yields

$$M = A_{4p+2}(2p+1)\kappa^{-1}A, \qquad (27)$$
$$Q = A_{4p+2}\mathcal{P}_0. \qquad (28)$$

up to some additive constants which can be are fixed to zero in order that the Minkowski space has vanishing charges.

III. DISCUSSION

In this paper we have presented an extension of the Maxwell action that enjoys the conformal symmetry in arbitrary dimension. We have take advantage of this symmetry to derive the equivalent of the electrically charged Reissner-Nordström black hole solutions in higher dimensions. These solutions only exist for dimensions which are multiples of four. The restriction on the dimension arises because we have restricted ourselves to the conformal case with a purely radial electric field. Using weaker hypothesis, in particular considering as a source an arbitrary power of the Maxwell invariant (not necessarily the conformal one), black holes solutions can be obtained [15] in any dimension.

The black holes presented here differ from the standard higher-dimensional solutions [16] since a) the spacetimes have vanishing scalar curvature, and b) the electric charge term in the metric coefficient goes as $r^{-(d-2)}$ and in the standard case is $r^{-2(d-3)}$.

As it is well-known, the Kerr-Newman metric represents the most general stationary, axisymmetric asymptotically flat solution of Einstein equations in the presence of an electromagnetic field in four dimensions. This spacetime geometry described a charged rotating black hole which reduces to the Reissner-Nordström solution at the vanishing angular momentum limit. It is natural to ask whether the extended Maxwell action considered here can act as a source for a Kerr-Newman like metric in dimensions higher than four.

The clue of the conformal invariance lies in the fact that we have considered power of the Maxwell invariant. This idea has been applied in the case of scalar field for which it has been shown that particular power of the massless Klein-Gordon Lagrangian exhibits conformal invariance in arbitrary dimension [17]. It would be interesting to see whether black hole solutions can also be obtained in this case. Empathizing on the conformal character of the matter source, one can consider as the conformal source of the Einstein equations in arbitrary dimension, the conformal electromagnetic action (1) together with a scalar field nonminimally and conformally coupled to gravity. This problem has already been solved in four dimensions [1]-[2] and, the solution has been shown to admit a generalization which possesses magnetic monopole [18]. One can also explore the possibility of considering as the conformal source, the conformal power of the Klein-Gordon action [17] together with the action (1).

Finally, it is also be desirable to study the geometric properties, the causal structures as well as the thermodynamics properties of the black hole solutions derived here.

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